SPECTRA OF INFINITE GRAPHS VIA SCHUR COMPLEMENT

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ABSTRACT. The goal of the paper is to apply the general operator theoretic construction known as the Schur complement for computation of the spectrum of certain infinite graphs which can be viewed as finite graphs with the ray attached to them. The examples of a multiple star and a flower with infinite rays are considered.

INTRODUCTION AND PRELIMINARIES

0.1. **Graph theory.** We begin with some basics of the graph theory. For the sake of simplicity we restrict ourselves with simple, connected, undirected, finite or infinite (countable) graphs, although the main result holds for weighted multigraphs and graphs with loops. We will label the vertex set $\mathcal{V}(\Gamma)$ by positive integers $\mathbb{N} = \{1, 2, \ldots\}$, $\{v\}_{v \in \mathcal{V}} = \{j\}_{j=1}^{\omega}$, $\omega \leq \infty$. The symbol $i \sim j$ means that the vertices i and j are incident, i.e., $\{i, j\}$ belongs to the edge set $\mathcal{E}(\Gamma)$.

The degree (valency) of a vertex $v \in \mathcal{V}(\Gamma)$ is a number $\gamma(v)$ of edges emanating from v. A graph Γ is said to be locally finite, if $\gamma(v) < \infty$ for all $v \in \mathcal{V}(\Gamma)$, and uniformly locally finite, if $\sup_{\mathcal{V}} \gamma(v) < \infty$.

The spectral graph theory studies the spectra and spectral properties of certain matrices related to graphs (more precisely, operators generated by such matrices in the standard basis $\{e_k\}_{k \in \mathbb{N}}$ and acting on the corresponding Hilbert spaces $\mathbb{C}^n$ or $\ell^2 = \ell^2(\mathbb{N})$). One of the most notable of them is the *adjacency matrix* $A(\Gamma)$

$$(0.1) \qquad A(\Gamma) = \|a_{ij}\|_{ij=1}^{\omega}, \quad a_{ij} = \begin{cases} 1, & \{i,j\} \in \mathcal{E}(\Gamma); \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding adjacency operator will be denoted by the same symbol. It acts as

$$(0.2) \qquad A(\Gamma)\, e_k = \sum_{j \sim k} e_j, \qquad k \in \mathbb{N}.$$

Clearly, $A(\Gamma)$ is a symmetric, densely-defined linear operator, whose domain is the set of all finite linear combinations of the basis vectors. The operator $A(\Gamma)$ is bounded and self-adjoint in ℓ^2, as long as the graph Γ is uniformly locally finite.

Whereas the spectral theory of finite graphs is very well established (see, e.g., [2, 3, 4]), the corresponding theory for infinite graphs is in its infancy. We refer to [9, 10, 14] for the basics of this theory. In contrast to the general consideration in [10], our goal is to compute the spectra of certain infinite graphs (precisely, the spectra of corresponding adjacency operators) which loosely speaking can be called "finite graphs with tails attached to them". To make the notion precise, we define first an operation of coupling well known for finite graphs (see, e.g., [4, Theorem 2.12]).

Definition 0.1. Let Γ_k, $k = 1, 2$, be two graphs with no common vertices, with the vertex sets and edge sets $\mathcal{V}(\Gamma_k)$ and $\mathcal{E}(\Gamma_k)$, respectively, and let $v_k \in \mathcal{V}(\Gamma_k)$. A graph $\Gamma = \Gamma_1 + \Gamma_2$ will be called a *coupling by means of the bridge* $\{v_1, v_2\}$ if

$$(0.3) \qquad \mathcal{V}(\Gamma) = \mathcal{V}(\Gamma_1) \cup \mathcal{V}(\Gamma_2), \qquad \mathcal{E}(\Gamma) = \mathcal{E}(\Gamma_1) \cup \mathcal{E}(\Gamma_2) \cup \{v_1, v_2\}.$$

So we join Γ_2 to Γ_1 by the new edge between v_2 and v_1.
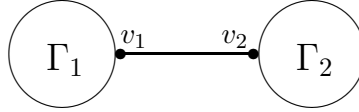
In general, the adjacency matrix $A(\Gamma)$ takes the form of a block matrix

$$(0.4) \qquad A(\Gamma) = \begin{bmatrix} A(\Gamma_1) & E \\ E & A(\Gamma_2) \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

If the graph Γ_1 is finite, $V(\Gamma_1) = \{1, 2, \dots, n\}$, and $V(\Gamma_2) = \{j\}_{j=1}^{\omega}$, we can with no loss of generality put $v_1 = 1$, $v_2 = n + 1$, so the adjacency matrix $A(\Gamma)$ can be written as the block matrix

$$(0.5) \qquad A(\Gamma) = \begin{bmatrix} A(\Gamma_1) & E_n \\ E_n & A(\Gamma_2) \end{bmatrix}, \qquad E_n = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots \end{bmatrix}.$$

If $\Gamma_2 = \mathbb{P}_{\infty}$, the one-sided infinite path, we can view the coupling $\Gamma = \Gamma_1 + \mathbb{P}_{\infty}$ as a finite graph with the tail. Now

$$(0.6) \qquad A(\Gamma_2) = J_0 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

is a Jacobi matrix called a *discrete Laplacian* or a *free Jacobi matrix*. It is of particular interest in the sequel.

The spectral theory of infinite graphs with one or several rays attached to certain finite graphs was initiated in [7, 8, 11] wherein a number of particular examples of graphs was examined. Our argument is based on a general construction from block operator matrices theory known as the Schur complement. As a matter of fact, the procedure applies not only to adjacency matrices, but to both Laplacians on graphs of such type.

The examples in the next section rely heavily on the formula of Schwenk [13] (see [4, Problem 2.7.9]) for characteristic polynomials of finite graphs F

$$P(\lambda, F) := \det(\lambda I - A(F)).$$

Given a graph F and $V \subset \mathcal{V}(F)$ we denote by $F\backslash V$ the subgraph of F induced by the vertices of $\mathcal{V}(F)\backslash V$.

Theorem (Schwenk). For a given finite graph F and $v \in \mathcal{V}(F)$, let $\mathcal{C}(v)$ denote the set of all simple cycles Z which contain v. Then

$$P(\lambda, F) = \lambda P(\lambda, F\backslash v) - \sum_{v' \sim v} P(\lambda, F\backslash\{v', v\}) - 2 \sum_{Z \in \mathcal{C}(v)} P(\lambda, F\backslash Z).$$

0.2. **Schur complement.** Let

$$(0.7) \qquad \mathcal{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be a block operator matrix which acts in orthogonal sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ of two Hilbert spaces. If A_{11} is invertible, the matrix $\mathcal{A}$ can be factorized as

$$(0.8) \qquad \mathcal{A} = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix},$$

I is the unity operator in the corresponding Hilbert space. Similarly, if A_{22} is invertible, one can write

$$(0.9) \qquad \mathcal{A} = \begin{bmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} C_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix}.$$

Here

$$(0.10) \qquad C_{22} := A_{22} - A_{21}A_{11}^{-1}A_{12}, \qquad C_{11} := A_{11} - A_{12}A_{22}^{-1}A_{21}$$

are usually referred to as the *Schur complements* [12]. Both equalities can be checked by multiplication.

The following result is a direct consequence of formulae (0.8) and (0.9).

Proposition 0.2. *Given a block operator matrix $\mathcal{A}$ (0.7), let A_{22} (A_{11}) be invertible. Then $\mathcal{A}$ is invertible if and only if so is C_{11} (C_{22}).*

Note that in the premises of Proposition 0.2 the inverse $\mathcal{A}^{-1}$ takes the form

$$\mathcal{A}^{-1} = \begin{bmatrix} C_{11}^{-1} & -C_{11}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}C_{11}^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}C_{11}^{-1}A_{12}A_{22}^{-1} \end{bmatrix}$$

and, respectively,

$$\mathcal{A}^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}C_{22}^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}C_{22}^{-1} \\ -C_{22}^{-1}A_{21}A_{11}^{-1} & C_{22}^{-1} \end{bmatrix}.$$

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It looks reasonable applying the latter result to

$$\lambda I - \mathcal{A} = \begin{bmatrix} \lambda I - A(\Gamma_1) & -E \\ -E & \lambda I - A(\Gamma_2) \end{bmatrix}, \qquad \lambda \in \mathbb{C},$$

to obtain

Proposition 0.3. *Given a block operator matrix $\mathcal{A}$ (0.7), let $\lambda \in \rho(A_{22})$ $\big(\lambda \in \rho(A_{11})\big)$. Then $\lambda \in \rho(\mathcal{A})$ if and only if $\lambda \in \rho(C_{11})$ $\big(\lambda \in \rho(C_{22})\big)$.*

Here, as usual, $\rho(T)$ stands for the resolvent set of a linear operator T, i.e., the set of complex numbers λ so that $\lambda I - T$ is boundedly invertible.

1. Spectra of infinite graphs with tails

We apply Proposition 0.3 to adjacency matrices (0.4) and (0.5) of couplings $\Gamma = \Gamma_1 + \Gamma_2$. It is clear now that

$$\lambda I - A(\Gamma) = (\lambda I - A(\Gamma_1)) \oplus (\lambda I - A(\Gamma_2)) + \Delta, \qquad \text{rank}(\Delta) = 2,$$

so we can gather some information about the spectrum of $A(\Gamma)$ for arbitrary graphs Γ_j, $j = 1, 2$, from the general perturbation theory of finite rank (see, e.g., [1, Section 9.2]). For instance, according to the famous result of Kato

$$\sigma_{ess}(A(\Gamma)) = \sigma_{ess}(A(\Gamma_1)) \cup \sigma_{ess}(A(\Gamma_2)),$$

(recall that a number $\lambda \in \sigma(T)$ belongs to the essential spectrum $\sigma_{ess}(T)$ of a self-adjoint operator T if it is not an eigenvalue of finite multiplicity). Moreover, λ is the eigenvalue of $A(\Gamma)$ as soon as λ is the isolated eigenvalue of either $A(\Gamma_1)$ or $A(\Gamma_2)$ of multiplicity at least 3. The Schur complements (0.10) take the form

(1.1) $\qquad C_{ii}(\lambda) = \lambda I - A(\Gamma_i) - G(\lambda, \Gamma_j)\, E, \qquad i \neq j, \quad i, j = 1, 2,$

where

(1.2) $\qquad G(\lambda, \Gamma_j) := \left((\lambda I - A(\Gamma_j))^{-1}\right)_{1,1}, \quad \lambda \in \rho(A(\Gamma_j)), \quad j = 1, 2$

is the so-called Green's function of $A(\Gamma_j)$.

We say that λ is a regular point of the graph Γ if $\lambda \in \rho(A(\Gamma))$.

Theorem 1.1. *Let $\Gamma = \Gamma_1 + \Gamma_2$ be the coupling of a finite graph Γ_1 and an arbitrary graph Γ_2, and let λ be a regular point of Γ_2. The point λ belongs to the spectrum of Γ if and only if it solves the equation*

(1.3) $\qquad\qquad P(\lambda, \Gamma_1) - G(\lambda, \Gamma_2)P(\lambda, \Gamma_1 \backslash v_1) = 0.$

Proof. By Proposition 0.3 and (1.1), $\lambda \in \sigma(A(\Gamma))$ if and only if

$$\det C_{11}(\lambda) = \det\big((\lambda I - A(\Gamma_1) - G(\lambda, \Gamma_2)\, E\big) = 0, \quad E = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

The result follows by expanding the latter determinant along the first row. $\qquad\square$

The basic example for us is $\Gamma_2 = \mathbb{P}_\infty$. In this case

$$\sigma(\Gamma) = \sigma_{ess}(\Gamma) \cup \sigma_d(\Gamma) = [-2, 2] \cup \sigma_d(\Gamma),$$

where the discrete spectrum $\sigma_d(\Gamma)$ is the set of eigenvalues of finite multiplicity off $[-2, 2]$. Theorem 1.1 can be applied, and the Green's function is known explicitly (see, e.g., [6])

$$(1.4) \quad \left(\lambda I - A(\mathbb{P}_\infty)\right)^{-1} = \|r_{ij}(z)\|_{i,j=1}^\infty, \quad r_{ij}(z) = \frac{z^{i+j} - z^{|i-j|}}{z - z^{-1}}, \quad \lambda = z + \frac{1}{z},$$

$|z| < 1$, and so

$$(1.5) \qquad\qquad G(\lambda, \mathbb{P}_\infty) = r_{11}(z) = z.$$

The discrete spectrum agrees with the zero set of algebraic equation (1.3)

$$(1.6) \quad \lambda \in \sigma_d(\Gamma) \Leftrightarrow P(\lambda, \Gamma_1) - xP(\lambda, \Gamma_1 \backslash v_1) = 0, \quad \lambda = x + \frac{1}{x}, \quad x \in (-1, 1).$$

So the problem amounts to computation of two characteristic polynomials and solving (1.6).

Example 1.2. "A multiple star".

Let $\kappa := (k_1, k_2, \ldots, k_n)$, $k_j \in \mathbb{N}$ be an n-tuple of positive integers. Denote by $S(\kappa)$ the graph obtained from the standard star graph $K_{1,n}$ with $n \geq 2$ edges by inserting $k_j - 1$ new vertices into j's edge, so this edge contains $k_j + 1$ vertices altogether. Put $\Gamma_1 = S(\kappa)$ and consider the coupling $\Gamma = \Gamma_1 + \mathbb{P}_\infty$ obtained from $S(\kappa)$ by attaching the infinite path to its root v_1.

Denote by $Q(\cdot, m)$ the characteristic polynomial of the finite path $\mathbb{P}_m$ with m vertices, so

$$Q(\lambda, m) = U_m\left(\frac{\lambda}{2}\right), \qquad U_m(\cos t) := \frac{\sin(m+1)t}{\sin t}$$

is the standard Chebyshev polynomial of the second kind.

To compute the characteristic polynomial of Γ_1 we apply the result of Schwenk which now looks

$$P(\lambda, \Gamma_1) = \lambda\, Q(\lambda) - Q(\lambda) \sum_{j=1}^n \frac{Q(\lambda, k_j - 1)}{Q(\lambda, k_j)}, \quad Q(\lambda) := \prod_{j=1}^n Q(\lambda, k_j).$$

It is clear that

$$P(\lambda, \Gamma_1 \backslash v_1) = Q(\lambda),$$

so the equation in (1.6) takes the form

$$Q(\lambda)\left\{\frac{1}{x} - \sum_{j=1}^n \frac{Q(\lambda, k_j - 1)}{Q(\lambda, k_j)}\right\} = 0, \quad \lambda = x + \frac{1}{x}.$$

Since $Q \neq 0$ off $[-2, 2]$ we come to

$$\sum_{j=1}^n \frac{Q(\lambda, k_j - 1)}{Q(\lambda, k_j)} - \frac{1}{x} = 0, \quad \lambda = x + \frac{1}{x}, \quad -1 < x < 1.$$

The function in the left-hand side is odd (as a function of x), so we can restrict ourselves with the values $0 < x < 1$. Putting $x = e^{-t}$, $t > 0$, we obtain after a bit of calculation

$$(1.7) \qquad \varphi(t) := \sum_{j=1}^{n} \frac{\sinh k_j t}{\sinh(k_j + 1)t} = e^t, \qquad t > 0.$$

Since the function

$$\frac{\sinh at}{\sinh bt}, \qquad 0 < a < b, \quad t > 0$$

is easily seen to be monotone decreasing (it follows, e.g., from the infinite product expansion of $\sinh z$), then so is φ in the left-hand side of (1.7). Moreover, φ vanishes at infinity. Next,

$$\lim_{t \to 0} \varphi(t) = \sum_{j=1}^{n} \frac{k_j}{k_j + 1} > 1$$

(we discard the trivial configuration $n = 2$, $k_1 = k_2 = 1$). So (1.7) has a unique solution $t_+ > 0$. Finally, the discrete spectrum is

$$(1.8) \qquad \sigma_d(\Gamma) = \pm\lambda_+, \qquad \lambda_+ := 2\cosh t_+.$$

In particular case $k_j = p$, $1 \le j \le n$, equation (1.7) looks

$$n\frac{\sinh pt}{\sinh(p+1)t} = e^t, \quad n(e^{pt} - e^{-pt}) = e^{(p+2)t} - e^{-pt},$$

which is equivalent to

$$(n-1)x^{2p+2} - nx^2 + 1 = 0.$$

Note that in this case the more detailed description of the spectrum is available, see [5, Example 2.3]. Precisely, there are p eigenvalues lying on the essential spectrum $[-2, 2]$.

Example 1.3. "A flower with n petals".

In this example Γ_1 is composed of n cycles $\{\mathbb{C}_j\}_{j=1}^{n}$, glued together at one common vertex (root) $\mathcal{O}$. Put $\Gamma = \Gamma_1 + \mathbb{P}_\infty$ with the infinite path attached to the root $\mathcal{O}$. Assume that the cycle $\mathbb{C}_j$ contains $k_j + 1$ vertices.

To compute the characteristic polynomial of Γ_1 we apply again the Schwenk theorem which gives

$$P(\lambda, \Gamma_1) = Q(\lambda)\left\{\lambda - 2\sum_{j=1}^{n}\frac{Q(\lambda, k_j - 1) + 1}{Q(\lambda, k_j)}\right\}.$$

As in the above example, $P(\cdot, \Gamma_1\backslash\mathcal{O}) = Q$, and we come to the following equation

$$2\sum_{j=1}^{n}\frac{Q(\lambda, k_j - 1) + 1}{Q(\lambda, k_j)} - \frac{1}{x} = 0, \quad \lambda = x + \frac{1}{x}, \quad -1 < x < 1.$$

Putting $x = e^{-t}$, $t > 0$, we obtain

$$(1.9) \qquad 2\sum_{j=1}^{n}\frac{\sinh k_j t}{\sinh(k_j + 1)t} = e^t, \qquad t > 0.$$

The same argument as above shows that (1.9) has a unique solution $t_+ > 0$, so the point $\lambda_+ = 2\cosh t_+ \in \sigma_d(\Gamma)$ for *all* configurations in Γ_1 (with no exceptions). Putting $x = -e^{-t}$, $t > 0$, we come to

$$(1.10) \qquad \varphi(t) := 2\sum_{j=1}^{n} \frac{\sinh k_j t + (-1)^{k_j+1}\sinh t}{\sinh(k_j+1)t} = e^t, \qquad t > 0.$$

Since

$$\frac{\sinh mt - \sinh t}{\sinh(m+1)t} = \frac{\sinh\frac{m-1}{2}t}{\sinh\frac{m+1}{2}t},$$

φ is a monotone decreasing function vanishing at infinity, and $\varphi(0+) > 1$ for all configurations in Γ_1. Hence (1.10) has a unique solution $t_- > 0$, and $\lambda_- = -2\cosh t_- \in \sigma_d(\Gamma)$. Finally, the discrete spectrum is

$$(1.11) \qquad \sigma_d(\Gamma) = \lambda_\pm, \qquad \lambda_\pm := \pm 2\cosh t_\pm.$$

Note that in particular case $n = 2$, $k_1 = k_2$ (the propeller with equal blades), a complete description of the spectrum is given in [5, Example 3.4].

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